

## A FORMULA FOR CASSON'S INVARIANT

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**ABSTRACT.** Suppose  $H$  is a homology 3-sphere obtained by Dehn surgery on a link  $L$  in a homology 3-sphere  $M$ . If every pair of components of  $L$  has zero linking number in  $M$ , then we give a formula for the Casson invariant,  $\lambda(H)$ , in terms of  $\lambda(M)$ , the surgery coefficients of  $L$ , and a certain coefficient from each of the Conway polynomials of  $L$  and all its sublinks. A few consequences of this formula are given.

**Introduction.** Andrew Casson has discovered an integral invariant of oriented homology 3-spheres which reduces, mod 2, to the Rochlin, or  $\mu$ -invariant [1]. In this paper we give a formula for Casson's invariant in the case where the homology sphere  $H$  is obtained by Dehn surgery on a link  $L$  in a homology sphere  $M$ , and furthermore the linking number between every pair of components of  $L$  is zero. (Note that every homology 3-sphere can be obtained in this way with  $M = S^3$ .) In this case the Casson invariant,  $\lambda(H)$ , can be expressed in terms of  $\lambda(M)$ , the surgery coefficients of  $L$ , and a certain coefficient from each of the Conway polynomials of  $L$  and all its sublinks.

To illustrate the utility of this formula, we compute the Casson invariant of homology spheres that arise as cyclic branched covers branched along untwisted double knots. We also describe how the  $\lambda$ -invariant changes when a crossing of the framed link  $L$  (between two strands of the same component) is changed. This provides, at least in this setting, an effective method for computing the invariant.

An interesting question is whether it is possible to show directly that the formula we give is preserved by the Kirby-Rolfsen calculus and thus give an alternative proof of the invariance of  $\lambda$ . If  $L_1$  and  $L_2$  are two (rationally) framed links representing the same manifold, then  $L_1$  can be transformed into  $L_2$  by a sequence of three types of moves: "homeomorphism," "twisting," and "trivial insertion or deletion" [8]. If the linking number between every pair of components of  $L_1$  is zero, then the first two types of moves will preserve this property while the third move may not. However, if  $L_1$  can be transformed into  $L_2$  through a sequence of links all of whose linking numbers are all zero, then we show that the formula for  $\lambda$  is preserved. We do not know if such a transformation is, in general, possible.

In order to state Casson's theorem and the formula for  $\lambda(H)$  we first establish some notation. Suppose  $L = \{K_1, \dots, K_n\}$  is a framed link in a 3-manifold  $M$  with

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rational framings  $p_1/q_1, \dots, p_n/q_n$ , respectively. We will denote by  $\chi(K_1, \dots, K_n; p_1/q_1, \dots, p_n/q_n; M)$ , or more simply by  $\chi(L; M)$ , the manifold obtained from  $M$  by Dehn surgery along  $L$  according to the given framings. If  $M$  is a homology sphere, then the Conway polynomial of  $L$  in  $M$ , which we will denote as  $\nabla_{L;M}(z)$ , has the form

$$\nabla_{L;M}(z) = z^{n-1}(a_0 + a_1 z^2 + \dots + a_m z^{2m}), \quad a_i \in \mathbb{Z}.$$

(This is well known if  $M = S^3$  and is shown, for example, in [4 or 7]. But given any link  $L$  in  $M$  it is easy to construct a link  $L'$  in  $S^3$  having the same number of components and same Conway polynomial as  $L$ .) Let  $\varphi_i(L; M) = a_i$  and let  $\text{lk}(K, J; M)$  denote the linking number of  $K$  and  $J$  in the homology sphere  $M$ . If  $M = S^3$  we will delete  $M$  from the notation.

**THEOREM (CASSON).** I. *There exists a unique integer invariant  $\lambda$  of oriented homology 3-spheres such that*

1.  $\lambda(S^3) = 0$ , and
2. *If  $K$  is a knot in a homology 3-sphere  $M$ , then*

$$\lambda(\chi(K; 1/(q+1); M)) - \lambda(\chi(K; 1/q; M)) = \varphi_1(K; M).$$

II. *The  $\lambda$ -invariant has the following properties:*

1.  $\lambda(-M) = -\lambda(M)$ .
2.  $\lambda(M) \neq 0 \Rightarrow$  *there exists a nontrivial representation of  $\pi_1(M)$  in  $SU(2)$ .*
3.  $\mu(M) \equiv \lambda(M) \pmod{2}$ , *where  $\mu(M)$  is the Rochlin invariant of  $M$ .*

From properties 1 and 2 of I, and induction on  $q$ , it follows that

$$\lambda(\chi(K; 1/q; M)) = \lambda(M) + q\varphi_1(K; M).$$

We will extend this formula to a link of  $n$  components, all of whose linking numbers are zero. Suppose  $L = \{K_1, \dots, K_n\}$  is such a link with framings  $1/q_i$ , respectively. We will show that

$$\lambda(\chi(L; M)) = \lambda(M) + \sum_{L' \subset L} \left( \prod_{i \in L'} q_i \right) \varphi_1(L'; M).$$

Here the sum is taken over all sublinks  $L'$  of  $L$  and the product over all  $i$  for which  $K_i$  is a component of  $L'$ . We have abbreviated this as  $i \in L'$ . Actually, the sum need only be taken over all sublinks having less than four components as  $\varphi_1(L'; M) = 0$  otherwise. (Notice that since the components of  $L$  do not link each other,  $\chi(L; M)$  is a homology sphere if and only if each surgery coefficient is of the form  $1/q_i$ .)

The only difficulty in deriving this formula from Casson's theorem is in computing  $\varphi_1(K_n; \chi(K_1, \dots, K_{n-1}; M))$  in terms of the original framed link data. §1 is devoted to doing this. Along the way we establish some facts for rationally framed links whose counterparts for integrally, or "honest", framed links are well known. Then in §2 we develop the formula for  $\lambda(\chi(L; M))$  and also show that it is preserved under the restricted Kirby-Rolfsen calculus described above. Finally, in §3, we discuss two applications. The first is the following theorem.

**THEOREM.** *Let  $K$  be a knot in the homology 3-sphere  $M$ ,  $DK$  its untwisted double, and  $\tilde{M}$  the  $p$ -fold cyclic branched cover of  $M$  branched along  $DK$ . Then  $\lambda(\tilde{M}) = p\lambda(M) + 2p\varphi_1(K; M)$ .*

The second application provides a means of recursively computing  $\lambda(H)$  by changing and smoothing crossings in a projection of some framed link  $L$  representing  $H$ . Again, the link  $L$  must have all linking numbers equal to zero.

**1. Linking in  $\chi(L; M)$ .** Suppose that  $L = \{K_1, \dots, K_n\}$  is a framed oriented link in a homology 3-sphere  $M$ , with framings  $p_1/q_1, \dots, p_n/q_n$ . The *linking matrix*  $\mathbf{B} = (b_{ij})$  associated to  $L$  is defined as follows. The off-diagonal entries are simply the linking numbers  $b_{ij} = \text{lk}(K_i, K_j; M)$ , while the diagonal entries are the surgery coefficients  $b_{ii} = p_i/q_i$ . If  $\bar{\mathbf{B}}$  is obtained from  $\mathbf{B}$  by multiplying the  $i$ th row by  $q_i$ , then it is easy to see that  $\bar{\mathbf{B}}$  is a presentation matrix for  $H_1(\chi(L; M); \mathbf{Z})$ . Hence  $\chi(L; M)$  is a homology sphere iff  $\det \bar{\mathbf{B}} = \pm 1$ , which is equivalent to  $\det \mathbf{B} = \pm 1/q_1 \cdots q_n$ . Suppose this is the case. Now if  $J_1$  and  $J_2$  are two knots in  $M - L$ , then we may think of them either as knots in  $M$  or as in  $\chi(L; M)$ . In either case they have a well-defined linking number and the purpose of our first lemma is to relate these two linking numbers. Let  $\text{lk}(J_i, L; M)$  be the  $1 \times n$  matrix whose  $j$ th entry is  $\text{lk}(J_i, K_j; M)$ .

**LEMMA 1.1.** *Suppose  $J_1$  and  $J_2$  are two knots in  $M - L$ . Then*

$$\text{lk}(J_1, J_2; \chi(L; M)) = \text{lk}(J_1, J_2; M) - \text{lk}(J_1, L; M)\mathbf{B}^{-1}\text{lk}(J_2, L; M)^T.$$

**PROOF.** Notice that even though  $\mathbf{B}$  is a matrix over  $\mathbf{Q}$ ,  $\mathbf{B}^{-1}$  has only integer entries! This is because  $\chi(L; M)$  is a homology sphere.

Suppose first that  $\text{lk}(J_1, L; M) = 0$ . Then  $J_1$  bounds a Seifert surface  $F$  in  $M - L$ . We may obtain  $F$  by starting with any Seifert surface  $F'$  in  $M$ . Now each  $K_i$  meets  $F'$  algebraically zero times. Hence we may remove disks from  $F'$  centered at  $K_i \cap F'$  and replace them with tubes that miss  $K_i$ . Now both  $\text{lk}(J_1, J_2; \chi(L; M))$  and  $\text{lk}(J_1, J_2; M)$  are given by the algebraic intersection of  $J_2$  with  $F$  and hence are equal.

If  $\text{lk}(J_1, L; M) \neq 0$ , then we will “slide”  $J_1$  over the components of  $L$  until this is the case. Since the reader is perhaps unfamiliar with this process, at least in the presence of rational surgery coefficients, we will briefly describe it.

If  $L$  has only integer framings  $p_i$ , then we may view  $\chi(L; M)$  as (one component of) the boundary of the 4-manifold obtained by attaching 2-handles to  $M \times I$  along  $L \subset M \times \{1\}$  according to the given framings. Now this description is easily altered by sliding the  $i$ th handle over the  $j$ th handle. The effect on the appearance of  $L$  is to replace  $K_i$  with some band connected sum of  $K_i$  with the “ $p_j$  push off” of  $K_j$ , that is, the curve, oriented parallel to  $K_j$ , which is the boundary of a disk in the  $j$ th 2-handle parallel to the core of the handle. The band connected sum may either respect or disrespect the orientations of the two curves, which corresponds to either “adding” or “subtracting” the handles. Now the linking matrix  $\mathbf{B}$  changes under this operation as

$$\mathbf{B} \rightarrow (\mathbf{I} \pm \mathbf{E}_{ij})\mathbf{B}(\mathbf{I} \pm \mathbf{E}_{ji}).$$

Here,  $\mathbf{E}_{ij}$  is the elementary matrix having a one in the  $(i, j)$  entry and zeros elsewhere. The “+” is used if the band connected sum respects orientations, the “−” otherwise.

Now if  $L$  has rational framings, we may no longer view  $\chi(L; M)$  as the boundary of a 4-manifold. But we may still alter  $L$  by replacing  $K_i$  with a band connected sum of  $K_i$  and the  $p_j/q_j$  push off of  $K_j$ . This simply amounts to isotoping  $K_i$  within  $\chi(L - K_i; M)$  before performing Dehn surgery on it. The  $p_j/q_j$  push off of  $K_j$  is the  $(q_j, p_j)$  cable of  $K_j$ , oriented parallel to  $K_j$ . If we adopt the convention that rational surgery coefficients are always reduced, and furthermore that the denominator is nonnegative, then  $\mathbf{B}$  changes as

$$\mathbf{B} \rightarrow (\mathbf{I} \pm q_j \mathbf{E}_{ij}) \mathbf{B} (\mathbf{I} \pm q_j \mathbf{E}_{ji}).$$

Here again, the “+” is used if the band connected sum respects orientations, “−” otherwise.

Returning to the proof of the lemma, we would like to slide  $J_1$  over the components of  $L$  until  $\text{lk}(J_1, L; M) = 0$ . Let  $\mathbf{A}$  be the linking matrix of  $L \cup J_1$ , where  $J_1$  has been given the unique framing that makes  $\det \mathbf{A} = 0$ . This corresponds to zero surgery on  $J_1$  in  $\chi(L; M)$ . (This choice of framing is merely a convenience.) We want to find a matrix  $\mathbf{P}$  such that  $\mathbf{P} = \prod (\mathbf{I} \pm q_i \mathbf{E}_{in+1})$  and moreover

$$(1.1) \quad \mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}.$$

To find  $\mathbf{P}$  we first find a vector  $\mathbf{v} = (v_1 \cdots v_n 1)^T$  such that  $\mathbf{A} \mathbf{v} = \mathbf{0}$ . We will then find  $\mathbf{P}$  so that  $\mathbf{P}(0 \cdots 0 1)^T = \mathbf{v}$ . Now

$$\mathbf{P}^T \mathbf{A} \mathbf{P} (0 \cdots 0 1)^T = \mathbf{P}^T \mathbf{A} \mathbf{v} = \mathbf{P}^T \mathbf{0} = \mathbf{0}$$

and hence equation (1.1) is true.

Let  $(v_1 \cdots v_n)^T = -\mathbf{B}^{-1} \text{lk}(J_1, L; M)^T$ . It is not hard to verify that this gives the desired vector  $\mathbf{v}$ . Now since  $\det \mathbf{B} = \pm 1/q_1 \cdots q_n$ , it follows that each  $v_i$  is divisible by  $q_i$ . Let  $v_i = q_i s_i$ . Then

$$\begin{aligned} \mathbf{P} &= \prod (\mathbf{I} + q_i s_i \mathbf{E}_{in+1}) = \prod (\mathbf{I} \pm q_i \mathbf{E}_{in+1})^{|s_i|} \\ &= \begin{bmatrix} \mathbf{I} & -\mathbf{B}^{-1} \text{lk}(J_1, L; M)^T \\ \mathbf{0} & 1 \end{bmatrix}. \end{aligned}$$

Hence we must slide  $J_1$  over  $K_i$   $|s_i|$  times.

We will now perform these slides in the presence of  $J_2$ . Let

$$\mathbf{P}' = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}' = \begin{bmatrix} \mathbf{B} & \text{lk}(J_1, L; M)^T & \text{lk}(J_2, L; M)^T \\ \text{lk}(J_1, L; M) & p/q & a \\ \text{lk}(J_2, L; M) & a & p'/q' \end{bmatrix},$$

where  $p'/q'$  is arbitrary. Then

$$\mathbf{P}'^T \mathbf{A}' \mathbf{P}' = \begin{bmatrix} \mathbf{B} & \mathbf{0} & \text{lk}(J_2, L; M) \\ \mathbf{0} & 0 & a' \\ \text{lk}(J_2, L; M) & a' & p'/q' \end{bmatrix}.$$

But now

$$\begin{aligned}
 \text{lk}(J_1, J_2; \chi(L; M)) &= a' = (0 \cdots 0 \ 1) \mathbf{P}'^T \mathbf{A}' \mathbf{P}' (0 \cdots 0 \ 1 \ 0)^T \\
 &= (0 \cdots 0 \ 1) \mathbf{A}' \left( (-\mathbf{B}^{-1} \text{lk}(J_1, L; M))^T \right)^T 1 \ 0)^T \\
 &= (\text{lk}(J_2, L; M) \ a \ p'/q') \left( (-\mathbf{B}^{-1} \text{lk}(J_1, L; M))^T \right)^T 1 \ 0)^T \\
 &= a - \text{lk}(J_2, L; M) \mathbf{B}^{-1} \text{lk}(J_1, L; M)^T. \quad \square
 \end{aligned}$$

Now suppose that  $K$  is a knot in  $M - L$  such that  $K$  bounds a Seifert surface  $F$  in  $M - L$ . Let  $e_1, \dots, e_r$  be a basis for  $H_1(F)$ . Now  $F$ , together with the choice of basis  $\{e_i\}$ , gives rise to two Seifert matrices: one for  $K$  considered as a knot in  $M$ , the other for  $K$  considered as a knot in  $\chi(L; M)$ . The  $(i, j)$  entry of the first matrix is given by  $\text{lk}(e_i^+, e_j; M)$ , and for the second by  $\text{lk}(e_i^+, e_j; \chi(L; M))$ . If we denote the first Seifert matrix as  $\mathbf{V}(K; M)$  and the second as  $\mathbf{V}(K; \chi(L; M))$ , then it follows easily from Lemma 1.1 that the two Seifert matrices are related as follows.

LEMMA 1.2. *Let  $M, L, K, F$ , and  $\{e_i\}$  be given as above. Then*

$$\mathbf{V}(K; \chi(L; M)) = \mathbf{V}(K; M) - \mathbf{E} \mathbf{B}^{-1} \mathbf{E}^T,$$

where  $\mathbf{E} = (e_{ij})$  is given by  $e_{ij} = \text{lk}(e_i, K_j; M)$ .

LEMMA 1.3. *Suppose  $\{K_1, \dots, K_n\}$  is an oriented link in a homology 3-sphere  $M$  such that  $\text{lk}(K_i, K_j; M) = 0$  for all  $i \neq j$ . Then there exist Seifert surfaces  $F_1$  and  $F_2$  such that  $\partial F_1 = K_1$ ,  $\partial F_2 = \{K_2, \dots, K_n\}$ , and  $F_1 \cap F_2$  is either empty or consists of a single ribbon intersection. Furthermore, in the latter case,  $F_1 \cap F_2 \subset \text{int } F_1$ ,  $F_1 \cap \partial F_2 \subset K_2$ , and  $F_1 \cap F_2$  does not separate  $F_2$ .*

PROOF. Let  $L = \{K_2, \dots, K_n\}$ . Since  $\text{lk}(K_i, K_j; M) = 0$  for all  $i \neq j$ , both  $K_1$  and  $L$  bound Seifert surfaces  $F_1$  and  $F_2$ , respectively, such that  $F_1 \cap L = \emptyset$  and  $F_2 \cap K_1 = \emptyset$ . Now put  $F_1$  and  $F_2$  in general position. Their intersection consists of a disjoint union of circles. If the intersection consists of only one circle, then we may isotope  $F_1$  as shown in Figure 1.1 to obtain new surfaces having a single ribbon intersection of the desired kind. If  $F_1 \cap F_2$  now separates  $F_2$ , then we may replace  $F_1$  with a surface that misses  $F_2$ .

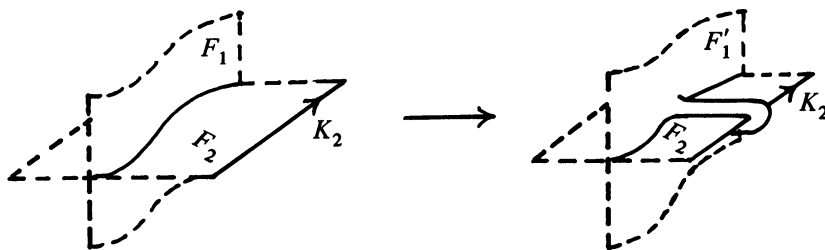


FIGURE 1.1

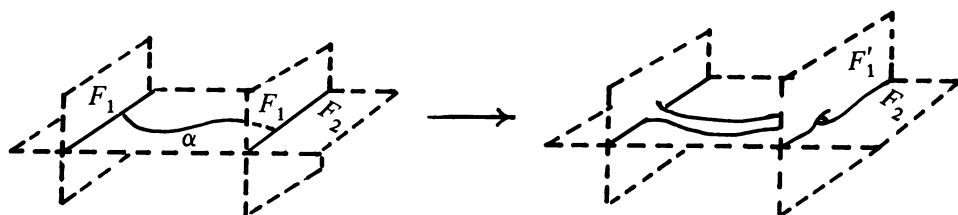


FIGURE 1.2

If the intersection contains more than one circle, then we may add a tube to  $F_1$ , as shown in Figure 1.2, provided the path  $\alpha$  in  $F_2$  that runs between the two curves of intersection, and which serves as the core of the tube, connects the same side of  $F_1$ . This will coalesce two circles of intersection into one. If it is not possible to find such an  $\alpha$  for two given circles  $c_1$  and  $c_2$  in  $F_1 \cap F_2$ , then we may first alter  $F_2$  by adding a tube  $T$  that connects the two sides of  $c_2$  and then letting  $\alpha$  run over this tube. This will always be possible, although we may need to first add tubes to  $F_2$  as shown in Figure 1.3.  $\square$

LEMMA 1.4. Let  $L = \{K_1, \dots, K_n\}$  be a link in a homology 3-sphere  $M$  with  $\text{lk}(K_i, K_j; M) = 0$  for all  $i \neq j$ . Suppose further that each  $K_i$  has framing  $1/q_i$ . Then for each  $1 \leq s \leq n$  we have

$$\begin{aligned} \varphi_1(K_n, \dots, K_s; \chi(K_{s-1}, \dots, K_1; M)) \\ = \sum_{L' \subset K_1, \dots, K_{s-1}} \left( \prod_{i \in L'} q_i \right) \varphi_1(L', K_s, \dots, K_n; M). \end{aligned}$$

Here the sum is taken over all sublinks of  $\{K_1, \dots, K_{s-1}\}$  including the empty sublink. The product is over all  $i$  such that  $K_i \subset L'$ , which we have abbreviated as  $i \in L'$ . If  $L'$  is empty we interpret the product as 1.

PROOF. We proceed by induction on  $n$ . If  $n = 1$ , then the formula is trivially true. So suppose that  $L$  is a link of  $n$  components but that the lemma is true for any link of  $n - 1$  or fewer components.

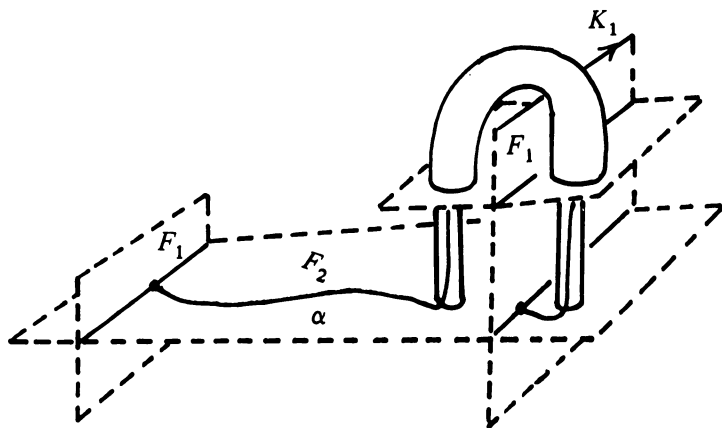


FIGURE 1.3

If  $s = 1$ , then again, the lemma is trivially true. So we shall begin with the case  $s = 2$ . Thus we seek to prove that

$$\varphi_1(K_2, \dots, K_n; \chi(K_1; M)) = \varphi_1(K_2, \dots, K_n; M) + q_1 \varphi_1(K_1, \dots, K_n; M).$$

Now by Lemma 1.3 there exist Seifert surfaces  $F_1$  and  $F_2$  such that  $\partial F_1 = K_1$ ,  $\partial F_2 = \{K_2, \dots, K_n\}$ , and either  $F_1 \cap F_2$  is empty or consists of a single ribbon intersection. If the intersection is empty, then  $\varphi_1(K_1, \dots, K_n; M) = 0$ . (Again, this is well known if  $M = S^3$ . See for example [6], and notice that the argument given there will work in the more general setting of an arbitrary homology sphere.) But by Lemma 1.2,  $\varphi_1(K_2, \dots, K_n; \chi(K_1; M)) = \varphi_1(K_2, \dots, K_n; M)$  since, for any choice of basis of  $H_1(F_2)$ ,  $\mathbf{E} = \mathbf{0}$ . Hence the lemma is true.

Now suppose that  $F_1 \cap F_2$  is a single ribbon intersection as described in Lemma 1.3. Let  $\{e_i\}$  be a basis for  $H_1(F_2)$  such that  $e_1$  meets  $F_1$  transversely in a single point and  $e_i \cap F_1 = \emptyset$  for  $i > 1$ . Hence  $\mathbf{E}^T = (\pm 1 \ 0 \ \dots \ 0)$  and by Lemma 1.2 we have

$$\mathbf{W} = \mathbf{V}(K_2, \dots, K_n; \chi(K_1; M)) = \mathbf{V}(K_2, \dots, K_n; M) - \mathbf{E}(q_1)\mathbf{E}^T.$$

By definition,  $\nabla_{K_2, \dots, K_n; \chi(K_1; M)}(z) = \det(t\mathbf{W} - t^{-1}\mathbf{W}^T)$ , where  $z = t - t^{-1}$ . This gives

$$\begin{aligned} \nabla_{K_2, \dots, K_n; \chi(K_1; M)}(z) &= \det(t\mathbf{V} - t^{-1}\mathbf{V}^T) - q_1 z \det(t\mathbf{V}_{11} - t^{-1}\mathbf{V}_{11}^T) \\ &= \nabla_{K_2, \dots, K_n; M}(z) - q_1 z \nabla_{L'; M}(z), \end{aligned}$$

where  $\mathbf{V} = \mathbf{V}(K_2, \dots, K_n; M)$ ,  $\mathbf{V}_{11}$  is the (1,1) minor of  $\mathbf{V}$ , and  $L'$  is the  $n$  component link that is spanned by the Seifert surface obtained by cutting  $F_2$  along  $F_1$ . Hence we have

$$\varphi_1(K_2, \dots, K_n; \chi(K_1; M)) = \varphi_1(K_2, \dots, K_n; M) - q_1 \varphi_0(L'; M).$$

Thus, it only remains to show that  $\varphi_1(K_1, \dots, K_n; M) = -\varphi_0(L'; M)$ .

Let  $F$  be a Seifert surface for the link  $L$  obtained from  $F_1$  and  $F_2$  as follows. Away from  $F_1 \cap F_2$  let  $F$  be  $F_1 \cup F_2$  and near the intersection let  $F$  appear as in Figure 1.4. Let  $\{d_j\}$  be a basis for  $H_1(F_1)$  so that  $\{c, \{d_j\}, \{e_i\}\}$  is a basis for  $H_1(F)$ , where  $c$  is the curve shown in the figure.

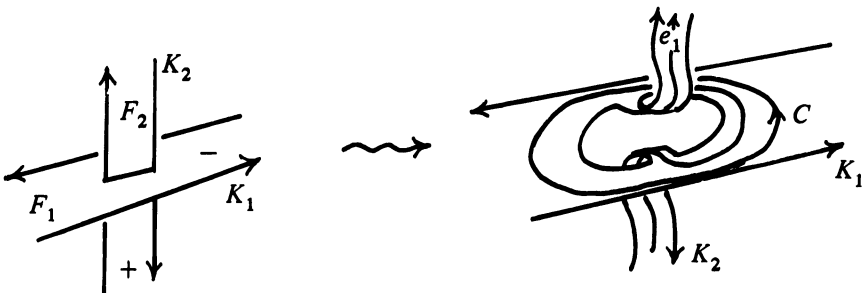


FIGURE 1.4

If  $\mathbf{V}' = \mathbf{V}(K_1; M)$  is the Seifert matrix determined by  $\{d_j\}$ , then a Seifert matrix for  $L$  in  $M$  has the form

$$\left[ \begin{array}{c|c|c} 0 & \mathbf{0} & 1 \ 0 \ \cdots \ 0 \\ \hline \mathbf{0} & \mathbf{V}' & \mathbf{A} \\ \hline 1 & & \\ 0 & & \\ \vdots & \mathbf{A}^T & \mathbf{V} \\ 0 & & \end{array} \right].$$

Hence we have

$$\begin{aligned} \nabla_{L;M}(z) &= \det \left[ \begin{array}{c|c|c} 0 & \mathbf{0} & z \ 0 \ \cdots \ 0 \\ \hline \mathbf{0} & t\mathbf{V}' - t^{-1}\mathbf{V}'^T & z\mathbf{A} \\ \hline z & & \\ 0 & & \\ \vdots & z\mathbf{A}^T & t\mathbf{V} - t^{-1}\mathbf{V}^T \\ 0 & & \end{array} \right] \\ &= -z^2 \det \left[ \begin{array}{cc} t\mathbf{V}' - t^{-1}\mathbf{V}'^T & z\mathbf{A}' \\ z\mathbf{A}'^T & t\mathbf{V}_{11} - t^{-1}\mathbf{V}_{11}^T \end{array} \right], \end{aligned}$$

where  $\mathbf{A}'$  is obtained from  $\mathbf{A}$  by removing the first column.

But since  $F_2$  is a surface with  $n - 1$  boundary components, we may assume that the  $\{e_i\}$  have been chosen so that  $\mathbf{V}$  has the form

$$\mathbf{V} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{D} \end{bmatrix},$$

where  $\mathbf{B}$  is a  $2h \times 2h$  matrix,  $\mathbf{C}$  is a  $2h \times n - 1$  matrix, and  $\mathbf{D}$  is an  $n - 1 \times n - 1$  symmetric matrix. This additional information gives

$$\nabla_{L;M}(z) = -z^2 \det \left[ \begin{array}{c|c} t\mathbf{V}' - t^{-1}\mathbf{V}'^T & z\mathbf{A}' \\ \hline z\mathbf{A}'^T & t\mathbf{B}_{11} - t^{-1}\mathbf{B}_{11}^T \quad z\mathbf{C}' \\ & z\mathbf{C}'^T \quad \mathbf{D} \end{array} \right],$$

where  $\mathbf{C}'$  is obtained from  $\mathbf{C}$  by deleting the first row.

Now  $\varphi_1(L; M)$  is the coefficient of  $z^{n+1}$  in  $\nabla_{L;M}(z)$ . This is actually the smallest power of  $z$  to appear since  $\text{lk}(K_i, K_j; M) = 0$  for all  $i \neq j$  implies that  $\varphi_0(L; M) = 0$ . Hence  $\nabla_{L;M}(z)/z^{n+1} = \varphi_1(L; M) + \varphi_2(L; M)z^2 + \cdots$ , and  $\varphi_1(L; M) = \lim_{z \rightarrow 0} \nabla_{L;M}(z)/z^{n+1}$ . But

$$\nabla_{L;M}(z)/z^{n+1} = -\det \left[ \begin{array}{c|c} t\mathbf{V}' - t^{-1}\mathbf{V}'^T & z\mathbf{A}' \\ \hline z\mathbf{A}'_1^T & t\mathbf{B}_{11} - t^{-1}\mathbf{B}_{11}^T \quad z\mathbf{C}' \\ \mathbf{A}'_2^T & \mathbf{C}'^T \quad \mathbf{D} \end{array} \right],$$



where  $A'_1$  is the first  $2h - 1$  columns of  $A'$  and  $A_2$  is the last  $n - 1$  columns. And so,

$$\begin{aligned}\varphi_1(L; M) &= -\det(V' - V'^T) \det \begin{bmatrix} B_{11} - B_{11}^T & 0 \\ C'^T & D \end{bmatrix} \\ &= -\nabla_{K_1; M}(0) \lim_{z \rightarrow 0} (\nabla_{L'; M}(z)/z^{n-1}) = -1 \cdot \varphi_0(L'; M).\end{aligned}$$

This completes the proof for  $s = 2$ .

Now assume that  $s > 2$ . We have, using our inductive hypothesis, that

$$\begin{aligned}\varphi_1(K_n, \dots, K_s; \chi(K_{s-1}, \dots, K_1; M)) \\ &= \varphi_1(K_n, \dots, K_s; \chi(K_{s-1}, \dots, K_2; \chi(K_1; M))) \\ &= \sum_{L'' \subset K_2, \dots, K_{s-1}} \left( \prod_{i \in L''} q_i \right) \varphi_1(L'', K_s, \dots, K_n; \chi(K_1; M)).\end{aligned}$$

Now, using the inductive hypothesis if  $L'' \neq \{K_2, \dots, K_{s-1}\}$  and the result for  $s = 2$  otherwise, we have

$$\begin{aligned}\varphi_1(K_n, \dots, K_s; \chi(K_{s-1}, \dots, K_1; M)) \\ &= \sum_{L'' \subset K_2, \dots, K_{s-1}} \left( \prod_{i \in L''} q_i \right) [\varphi_1(L'', K_s, \dots, K_n; M) \\ &\quad + q_1 \varphi_1(K_1, L'', K_s, \dots, K_n; M)] \\ &= \sum_{L'' \subset K_1, \dots, K_{s-1}} \left( \prod_{i \in L''} q_i \right) \varphi_1(L'', K_s, \dots, K_n; M). \quad \square\end{aligned}$$

Actually, many terms in the sum given in Lemma 1.4 are zero. This follows from the following lemma.

**LEMMA 1.5.** *Suppose  $L = \{K_1, \dots, K_n\}$  is a link in a homology sphere  $M$  with  $\text{lk}(K_i, K_j; M) = 0$  for all  $i \neq j$ , and furthermore  $n > 3$ . Then  $\varphi_1(L; M) = 0$ .*

**PROOF.** Let  $F_1$  and  $F_2$  be as in the proof of Lemma 1.4. As before, if  $F_1 \cap F_2 = \emptyset$ , then  $\varphi_1(L; M) = 0$ . If not, then we showed that  $\varphi_1(L; M) = -\varphi_0(L'; M)$ , where  $L'$  is the  $n$  component link that is spanned by the Seifert surface obtained by cutting  $F_2$  along  $F_1$ . But now  $\varphi_0(L'; M)$  depends only on the linking numbers of  $L'$  and can be computed as follows (see [3 or 5]). Let  $L' = \{J_1, \dots, J_n\}$ , where  $K_2 = J_1 \#_b J_2$  and  $K_i = J_i$  for  $i > 2$ . Let  $\mathcal{L} = (l_{ij})$  be the matrix given by  $l_{ij} = \text{lk}(J_i, J_j; M)$  if  $i \neq j$  and  $l_{ii} = -\sum_{j \neq i} l_{ij}$ . Then  $\varphi_0(L'; M)$  is equal to any cofactor of  $\mathcal{L}$ . But the  $(1, 1)$  cofactor of  $\mathcal{L}$  is clearly zero.  $\square$

Hence the sum given in Lemma 1.4 may actually just be taken over all 1, 2, and 3-component sublinks of  $L$ .

**2. A formula for  $\lambda$ .** In this section we will establish the formula for  $\lambda$  given in the Introduction. It is easily derived from Casson's theorem and Lemma 1.4.

**THEOREM 2.1.** *Let  $L = \{K_1, \dots, K_n\}$  be an oriented framed link in the homology 3-sphere  $M$  with framings  $1/q_1, \dots, 1/q_n$ , respectively. Furthermore, suppose that  $\text{lk}(K_i, K_j; M) = 0$  for all  $i \neq j$ . Then the Casson invariant of  $\chi(L; M)$  is given by*

$$\lambda(\chi(L; M)) = \lambda(M) + \sum_{L' \subset L} \left( \prod_{i \in L'} q_i \right) \varphi_1(L'; M).$$

*Actually, the sum need only be taken over those sublinks of  $L$  having less than four components.*

**PROOF.** We proceed by induction on  $n$ . The case when  $n = 1$  is a direct consequence of Casson's theorem as mentioned in the Introduction. Now if  $n > 1$  then by the induction hypothesis, as well as Lemma 1.4, we have

$$\begin{aligned} \lambda(\chi(L; M)) &= \lambda(\chi(K_n; \chi(K_1, \dots, K_{n-1}; M))) \\ &= \lambda(\chi(K_1, \dots, K_{n-1}; M)) + q_n \varphi_1(K_n; \chi(K_1, \dots, K_{n-1}; M)) \\ &= \lambda(M) + \sum_{L' \subset K_1, \dots, K_{n-1}} \left( \prod_{i \in L'} q_i \right) \varphi_1(L'; M) \\ &\quad + q_n \sum_{L' \subset K_1, \dots, K_{n-1}} \left( \prod_{i \in L'} q_i \right) \varphi_1(L', K_n; M) \\ &= \lambda(M) + \sum_{L' \subset L} \left( \prod_{i \in L'} q_i \right) \varphi_1(L'; M). \end{aligned}$$

Finally, using Lemma 1.5, we see that only sublinks having less than four components will contribute to the sum.  $\square$

It is not too difficult to see that the formula given in Theorem 2.1 does indeed reduce mod 2 to the  $\mu$ -invariant. For example, if  $M = S^3$  the formula reduces mod 2 to  $\sum_{L' \subset L''} \varphi_1(L')$ , where  $L''$  is the sublink of  $L$  for which each  $q_i$  is odd. But in [4 and 7] it is shown that this sum is congruent to the Arf invariant of  $L''$ . Now the original argument given by González in [2], in which he shows that

$$\mu(\chi(K; 1/q)) \equiv q \text{Arf}(K) \pmod{2},$$

can easily be extended to show that  $\mu(\chi(L)) \equiv \text{Arf}(L'') \pmod{2}$ .

If the orientation of  $M$  is reversed, then it is also easy to see that the formula for  $\lambda$  changes sign. For suppose  $M = \chi(L)$ . Then  $-M = \chi(rL)$ , where  $rL$  is the reflection of  $L$  (obtained by changing all the crossings in some projection of  $L$ ) and the framings of  $rL$  are the negations of the corresponding framings of  $L$ . Now it is well known that  $\nabla_{rL}(z) = \nabla_L(-z)$ . Hence  $\varphi_1(rL) = \varphi_1(L)$  if  $L$  has an odd number of components and  $\varphi_1(rL) = -\varphi_1(L)$  otherwise. Therefore, each term in the formula for  $\lambda(\chi(L))$  is negated when  $L$  is replaced with  $rL$ .

In the remainder of this section we will show that the formula for  $\lambda$  is preserved by the Kirby-Rolfsen calculus, provided the condition that all linking numbers are zero is maintained.

If  $L$  and  $L'$  are two rationally framed links in  $S^3$  such that  $\chi(L) = \chi(L')$ , then  $L$  can be transformed into  $L'$  by a finite sequence of three types of moves: homeomorphism, twisting, and trivial insertion or deletion [8]. A homeomorphism alters  $L$  by

replacing it with  $h(L)$ , where  $h: S^3 \rightarrow S^3$  is an orientation-preserving homeomorphism. The surgery coefficients of  $L$  are unchanged. Hence a move of this type clearly maintains all zero linking numbers and preserves the formula for  $\lambda$ . A trivial insertion or deletion amounts to inserting or deleting a component of  $L$  with framing  $\infty = 1/0$ . Even if such an insertion introduces nonzero linking numbers, the formula is still preserved by virtue of the  $\infty$  framing. Finally, consider altering  $L$  by means of twisting. This involves cutting the exterior of  $L$  open along a disk, twisting the now opposing disks through  $2\pi t$  radians, and then gluing the disks back together. Such a move is illustrated in Figure 2.1 where the disk spans the first component  $K_1$  of  $L$ . Let  $L'$  be the resulting framed link with  $K'_i$  the image of  $K_i$ .

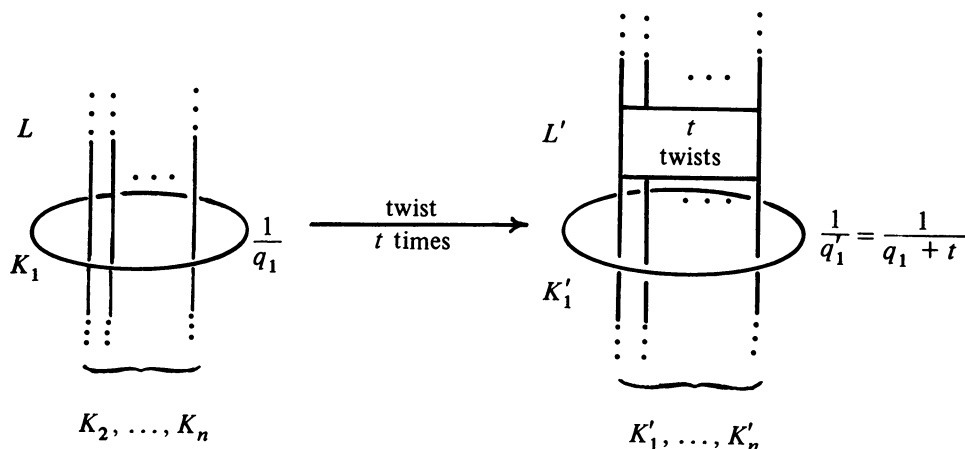


FIGURE 2.1

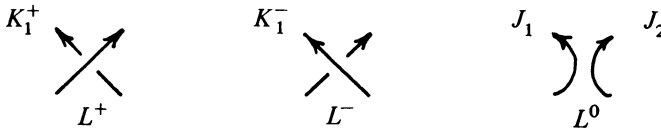
The surgery coefficients change as a result of the twisting with  $1/q'_1 = 1/(q_1 + t)$  and  $1/q'_i = 1/q_i$  for  $i > 1$  since all the linking numbers are assumed to be zero. Now  $\chi(K_1) \cong \chi(K'_1)$ , the homeomorphism being given by the twist. Hence if  $L$  is a link in  $\chi(K_1)$  and  $L'$  is its image in  $\chi(K'_1)$  under the homeomorphism, we clearly have that  $\varphi_1(L; \chi(K_1)) = \varphi_1(L'; \chi(K'_1))$ . This constitutes the principal step in the following sequence of equalities:

$$\begin{aligned}
 \sum_{L'' \subset L} \left( \prod_{i \in L''} q_i \right) \varphi_1(L'') &= \sum_{L'' \subset K_2 \cdots K_n} \left( \prod_{i \in L''} q_i \right) [\varphi_1(L'') + q_1 \varphi_1(K_1, L'')] \\
 &= \sum_{L'' \subset K_2 \cdots K_n} \left( \prod_{i \in L''} q_i \right) \varphi_1(L''; \chi(K_1)) \\
 &= \sum_{L'' \subset K'_2 \cdots K'_n} \left( \prod_{i \in L''} q'_i \right) \varphi_1(L''; \chi(K'_1)) \\
 &= \sum_{L'' \subset K'_2 \cdots K'_n} \left( \prod_{i \in L''} q'_i \right) [\varphi_1(L'') + q'_1 \varphi_1(K'_1, L'')] \\
 &= \sum_{L'' \subset L'} \left( \prod_{i \in L''} q'_i \right) \varphi_1(L'').
 \end{aligned}$$

**3. Applications.** In order to make use of Theorem 2.1 it is necessary to know  $\varphi_1(L')$  for each sublink  $L'$  of  $L$ . But to compute  $\nabla_{L'}(z)$  for each sublink of  $L$  is laborious and unnecessary. Instead, we can use the following theorem to compute  $\lambda(\chi(L))$ . It allows us to unknot a component of  $L$  by successively changing crossings in some projection. After the component is unknotted it can be “blown down,” that is, twisted until its framing is  $\infty$  and then deleted. This leaves a framed link with one fewer component and the process may then be repeated.

**THEOREM 3.1.** Suppose  $L^+ = \{K_1^+, K_2, \dots, K_n\}$ ,  $L^- = \{K_1^-, K_2, \dots, K_n\}$ , and  $L^0 = \{J_1, J_2, K_2, \dots, K_n\}$  are three oriented link projections such that

1. All linking numbers of  $L^+$  and  $L^-$  are zero.
2.  $L^+$ ,  $L^-$  and  $L^0$  are identical except near a single crossing of  $L^+$  where they appear instead as:



3. In all three projections  $K_i$  has framing  $1/q_i$  for  $i > 1$  and in  $L^+$  and  $L^-$ ,  $K_1$  has framing  $1/q_1$ . The knots  $J_1$  and  $J_2$  are unframed.

Then

$$\begin{aligned}\lambda(\chi(L^+)) - \lambda(\chi(L^-)) &= q_1 \text{lk}(J_1, J_2; \chi(K_2, \dots, K_n)) \\ &= q_1 [\text{lk}(J_1, J_2) - \sum q_i \text{lk}(J_1, K_i) \text{lk}(J_2, K_i)].\end{aligned}$$

**PROOF.** Suppose  $P^+$  is any sublink of  $L^+$ . Let  $P^-$  and  $P^0$  be the corresponding sublinks of  $L^-$  and  $L^0$  that result from changing and smoothing the crossing. If  $K_1^+$  is not a component of  $P^+$ , then  $P^+ = P^- = P^0$ . Otherwise we have

$$\nabla_{P^+}(z) - \nabla_{P^-}(z) + z \nabla_{P^0}(z) = 0.$$

From this it follows that

$$\varphi_1(P^+) - \varphi_1(P^-) + \varphi_0(P^0) = 0.$$

Now,

$$\begin{aligned}\lambda(\chi(L)) &= \sum_{L' \subset L} \left( \prod_{i \in L'} q_i \right) \varphi_1(L') \\ &= \sum_{L' \subset K_2, \dots, K_n} \left( \prod_{i \in L'} q_i \right) \varphi_1(L') + q_1 \left( \prod_{i \in L'} q_i \right) \varphi_1(K_1^+, L') \\ &= \sum_{L' \subset K_2, \dots, K_n} \left( \prod_{i \in L'} q_i \right) \varphi_1(L') + q_1 \left( \prod_{i \in L'} q_i \right) [\varphi_1(K_1^-, L') - \varphi_0(J_1, J_2, L')] \\ &= \lambda(\chi(L^-)) - q_1 \sum_{L' \subset K_2, \dots, K_n} \left( \prod_{i \in L'} q_i \right) \varphi_0(J_1, J_2, L').\end{aligned}$$

So it remains to prove that

$$\sum_{L' \subset K_2, \dots, K_n} \left( \prod_{i \in L'} q_i \right) \varphi_0(J_1, J_2, L') = -\text{lk}(J_1, J_2; \chi(K_2, \dots, K_n)).$$

But from Lemma 1.1 it follows that

$$\text{lk}(J_1, J_2; \chi(K_2, \dots, K_n)) = \text{lk}(J_1, J_2) - \sum q_i(J_1, K_i) \text{lk}(J_2, K_i).$$

Now again using the result in [3 or 5] that was used in the proof of Lemma 1.5, we have  $\varphi_0(J_1, J_2, L') = 0$  if  $L'$  has two or more components. Furthermore,

$$\begin{aligned} \varphi_0(J_1, J_2, K_i) &= \text{lk}(J_1, K_i) \text{lk}(J_2, K_i) + \text{lk}(J_1, J_2) [\text{lk}(J_1, K_i) + \text{lk}(J_2, K_i)] \\ &= \text{lk}(J_1, K_i) \text{lk}(J_2, K_i) + \text{lk}(J_1, J_2) \text{lk}(K_1^+, K_i) \\ &= \text{lk}(J_1, K_i) \text{lk}(J_2, K_i). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{L' \subset K_2, \dots, K_n} \left( \prod_{i \in L'} q_i \right) \varphi_0(J_1, J_2, L') &= \varphi_0(J_1, J_2) + \sum q_i \varphi_0(J_1, J_2, K_i) \\ &= \varphi_0(J_1, J_2) + \sum q_i \text{lk}(J_1, K_i) \text{lk}(J_2, K_i) \\ &= -\text{lk}(J_1, J_2; \chi(K_2, \dots, K_n)). \end{aligned}$$

Here we have used the fact that  $\varphi_0(J_1, J_2) = -\text{lk}(J_1, J_2)$ , which again follows from [3 or 5].  $\square$

To illustrate the usefulness of Theorem 2.1 we will prove the following theorem.

**THEOREM 3.2.** *Let  $K$  be a knot in the homology 3-sphere  $M$ ,  $DK$  its untwisted double, and  $\tilde{M}$  the  $p$ -fold cyclic branched cover of  $M$  branched along  $DK$ . Then  $\lambda(\tilde{M}) = p\lambda(M) + 2p\varphi_1(K; M)$ .*

**PROOF.** We will obtain a framed link description of both  $M$  and  $\tilde{M}$  and then use Theorem 2.1. To begin, suppose  $K \subset M = \chi(L) = \chi(K_1, \dots, K_n)$ . We may assume that  $\text{lk}(K_i, K_j) = 0$  for all  $i \neq j$ , that each  $K_i$  is an unknot with surgery coefficient  $1/q_i$ , and further that  $K \subset S^3 - L$  with  $\text{lk}(K, K_i) = 0$  for all  $i$ . Now we may untie  $K$  in  $S^3$  by changing crossings. If we introduce one surgery curve  $C_i$  for each crossing that is changed, we may arrive at a surgery description of  $K$  in  $M$  as shown in Figure 3.1. Inside the torus  $T$  lie  $K_1, \dots, K_n$  which represent  $M$  and also  $C_1, \dots, C_m$  which were used to untie  $K$ . We may assume each  $C_i$  is unknotted, links  $K$  zero times, has surgery coefficient  $\varepsilon_i = \pm 1$ , and, furthermore,  $\{C_i\}$  is an unlink when viewed either in  $S^3$  or  $M$ .

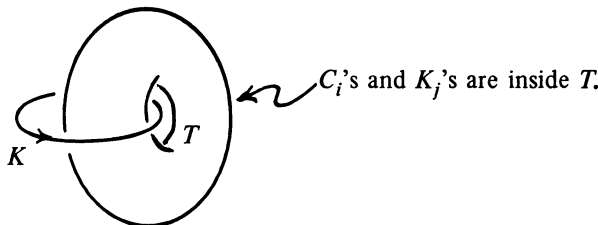


FIGURE 3.1

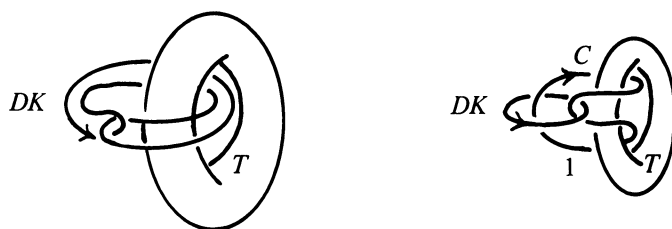


FIGURE 3.2

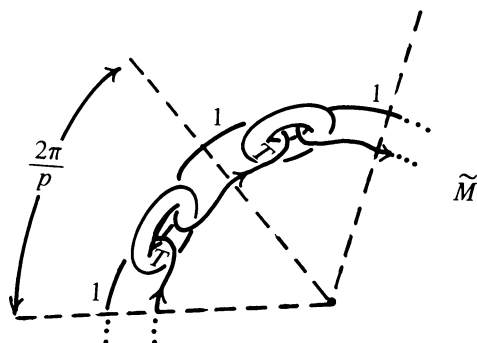


FIGURE 3.3

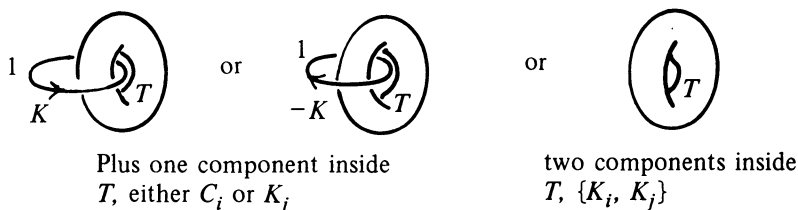


FIGURE 3.4

Now the untwisted double of  $K$  (with left-handed clasp), together with a surgery description of it, appears as in Figure 3.2. Finally in Figure 3.3 we have a surgery description of  $\tilde{M}$ . The surgery coefficients of the lifts of the  $C_i$ 's and  $K_j$ 's are the same upstairs as downstairs.

Let  $\tilde{L}$  be the link shown in Figure 3.3. We are concerned with the 1, 2, and 3-component sublinks of  $\tilde{L}$ . Now the components of  $\tilde{L}$  are all unknots and so do not contribute to  $\lambda(\tilde{M})$ . The two component sublinks of  $\tilde{L}$  which are not split (and hence have trivial Conway polynomial) are shown in Figure 3.4. In the case of two components inside the same lift of  $T$ , we may assume they are of the form  $\{K_i, K_j\}$  since the other possibilities are split. The three component sublinks that we must consider are shown in Figure 3.5. However, the sublinks shown in (a) are all boundary links since each  $K_i$  or  $C_i$  links  $K$  zero times. Hence they do not contribute to  $\lambda(\tilde{M})$ . Also the sublinks shown in (c) do not contribute. This is because each can be viewed as a connected sum of two links each having  $\varphi_0 = 0$ . Since the Conway

polynomial multiplies under a connected sum, we see that  $\varphi_1 = 0$  for such a link. Finally in case (d), we may assume the link is  $\{K_i, K_j, K_k\}$  since links involving the  $C_i$ 's are split.

Therefore we have

$$\begin{aligned}\lambda(\tilde{M}) = & p \sum q_i q_j \varphi_1(K_i, K_j) + p \sum q_i \varphi_1(K, K_i) + p \sum \varepsilon_j \varphi_1(K, C_j) \\ & + p \sum q_i \varphi_1(-K, K_i) + p \sum \varepsilon_j \varphi_1(-K, C_j) \\ & + p \sum q_i q_j \varphi_1(K, K_i, K_j) + p \sum q_i \varepsilon_j \varphi_1(K, K_i, C_j) + p \sum \varepsilon_i \varepsilon_j \varphi_1(K, C_i, C_j) \\ & + p \sum q_i q_j \varphi_1(-K, K_i, K_j) + p \sum q_i \varepsilon_j \varphi_1(-K, K_i, C_j) \\ & + p \sum \varepsilon_i \varepsilon_j \varphi_1(-K, C_i, C_j) + p \sum q_i q_j q_k \varphi_1(K_i, K_j, K_k).\end{aligned}$$

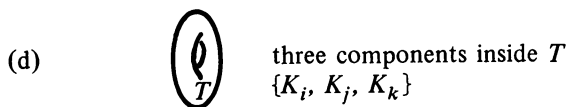
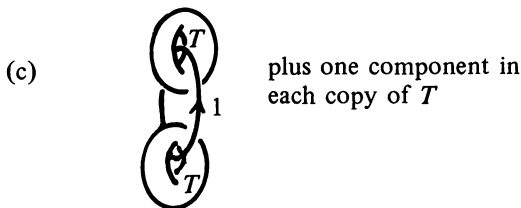
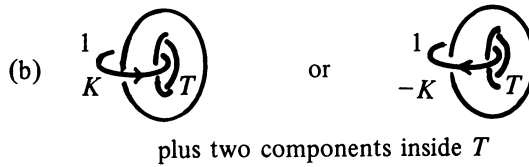
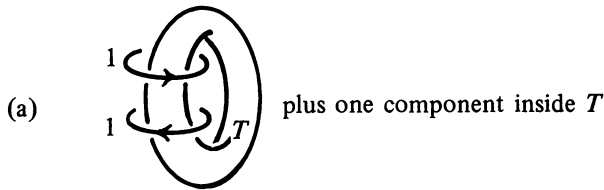


FIGURE 3.5

But computing  $\lambda(\chi(K, +1; M))$  in two different ways, we obtain

$$\begin{aligned}\lambda(M) + \varphi_1(K; M) &= \sum q_i \varphi_1(K, K_i) + \sum \varepsilon_j \varphi_1(K, C_j) + \sum q_i q_j \varphi_1(K_i, K_j) \\ &\quad + \sum q_i q_j \varphi_1(K, K_i, K_j) + \sum q_i \varepsilon_j \varphi_1(K, K_i, C_j) + \sum \varepsilon_i \varepsilon_j \varphi_1(K, C_i, C_j) \\ &\quad + \sum q_i q_j q_k \varphi_1(K_i, K_j, K_k).\end{aligned}$$

We may also replace  $K$  with  $-K$  in the above formula, but notice that  $\varphi_1(-K; M) = \varphi_1(K; M)$ .

Finally, applying Theorem 2.1 to  $L$  gives

$$\lambda(M) = \sum q_i q_j \varphi_1(K_i, K_j) + \sum q_i q_j q_k \varphi_1(K_i, K_j, K_k).$$

Combining these last two equations with the first gives the desired result.  $\square$

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